14 Example calculations

14.1 Problem 1: First order capacity, temperature

Hot water is produced in an open vessel, in a continuous process, by direct steam heating.

In the initial, steady, state the water is warmed up from $T_1(t=0) = 15°C$ to $T_2(t=0) = 75°C$ by $m_S = 30$ kg/h steam. After suddenly closing the steam valve, the outlet temperature of the water decreases according to the following function of time:

t [min]	0	2	4	8	12	16	20	25	30
T_2 [^{o}C]	75	69.5	64.6	56	48.9	43	38.1	33.2	29.4

Consider this as a process with m_S [kg/h] manipulated variable to T_2 [^{o}C] controlled variable.

Questions

a. Prove that this is a first order lag.

b. What are the dynamic parameters of this process?

c. After reaching a new stationary state with closed steam valve, we open the valve again so much that the new steam flow rate is $m_S = 50$ kg/h. How does the hot water temperature (T_2 [^{o}C]) changes is time, i.e. what is the function $T_2(t)$ [^{o}C]?

Solution

a. Prove that this is a first order lag.

We have to calculate logarithm of Ratio=$\frac{\hat{T}_2(\infty)-\hat{T}_2(t)}{\hat{T}_2(\infty)}$.

For this aim first we need $T_2(t=\infty)$. This is easily found as 15°C because this is the inlet temperature, and without heating it remains so. Thus $T_2(t=\infty) = 15°C$ and the change is $\hat{T}_2(\infty) = T_2(\infty) - T_2(0)$ = 15°C – 75°C = –60°C.

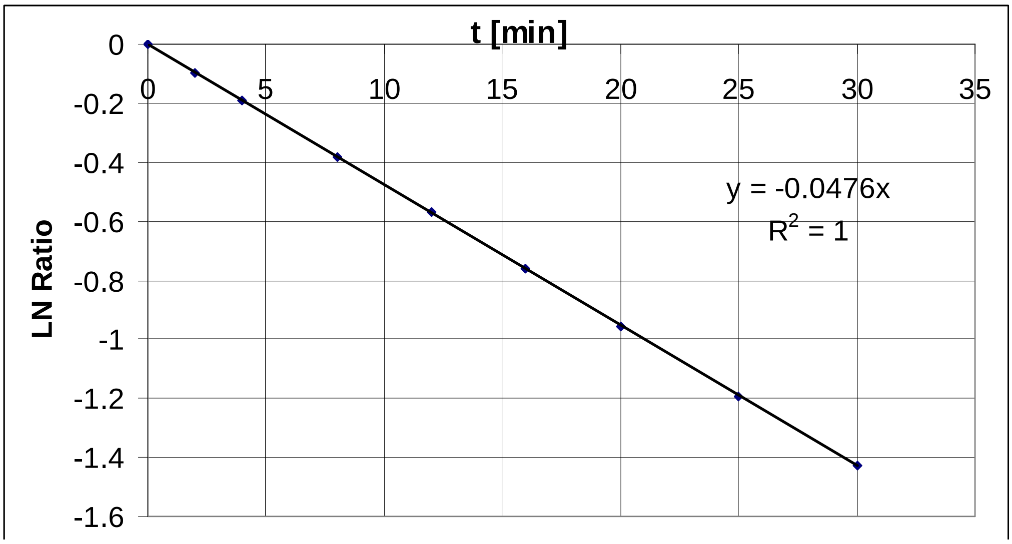
In the same way, $\hat{T}_2(t) = T_2(t) - T_2(0) = T_2(t) -$ 75°C.

This is how the calculation goes:

t [min]	0	2	4	8	12	16	20	25	30
T_2 [^{o}C] (measured)	75	69.5	64.6	56	48.9	43	38.1	33.2	29.4
$\hat{T}_2(t)$ [^{o}C]	0	-5.5	-10.4	-19	-26.1	-32	-36.9	-41.8	-45.6
$\hat{T}_2(\infty)$ [^{o}C]	–60	–60	–60	–60	–60	–60	–60	–60	–60
Ratio	1	0.908	0.827	0.683	0.565	0.467	0.385	0.3037	0.24
ln(Ratio)	0	–0.096	–0.19	–0.381	–0.571	–0.762	–0.955	–1.193	–1.427

One can see that the points lie along a straight line but this can also be proven by optimally fitting a straight line to the point (regression), see the next figure. The correlation coefficient R is 1; that means an almost perfect fit.

b. What are the dynamic parameters of this process?
This can be answered by finding the parameters of the straight line through optimal regression as is shown in the figure.



According to the fitting, the slope is –0.0476.

According to the theory, $\ln\frac{\hat{T}_2(\infty)-\hat{T}_2(t)}{\hat{T}_2(\infty)}=-\frac{1}{T}\cdot t$ where T is the time constant. From here the time constant T=1/0.0476=21 minutes.

The gain can be determined from a step response. Such a step disturbance was closing the valve: with a step value a=–30 kg/h. As a result, the final change in T_2 was $A\cdot a$=–60°C. From here the process gain is A=–60°C/–30 kg/h = 2°C/(kg/h).

c. After reaching a new stationary state with closed steam valve we open the valve again so much that the new steam flow rate is m_S = 50 kg/h. How does the hot water temperature (T_2 [°C]) changes is time, i.e. what is the function $T_2(t)$ [°C]?

In the new steady state before opening the valve $T_2(0)$ = 15°C. A step disturbance is applied to the steam mass flow rate, a = 50 kg/h. The step response of a first order lag:

$$\hat{T}_2(t)=a\cdot A\cdot\left[1-e^{-\frac{t}{T}}\right],\qquad T_2(t)=T_2(0)+a\cdot A\cdot\left[1-e^{-\frac{t}{T}}\right]$$

$$T_2(t)=15°C+50\frac{kg}{h}\cdot 2\frac{°C}{kg/h}\cdot\left[1-e^{-\frac{t}{21min}}\right]=15°C+100°C\cdot\left[1-e^{-\frac{t}{21min}}\right]$$

However, this function is valid under the boiling point only, i.e. up to 100°C, after that point the temperature does not increase but the water boils. For calculating the time of reaching this temperature we substitute:

$$100°C=15°C+100°C\cdot\left[1-e^{-\frac{t}{21min}}\right]$$, and from here: t=39.8 min.

That is, the function above is valid up to 39.8 min, and then it remains 100°C.

14.2 Problem 2: Capacities, heating, thermometer

W_w=500 l/h hot water is produced continuously in a perfectly mixed vessel by direct steam heating. In the initial, steady, state the water is warmed up from $\vartheta_w(t=0)$ = 20°C to $\vartheta_V(t=0)$ = 85°C and the steam flow rate is m_S = 60 kg/h.

Heat capacity of the vessel (of the process material resided in the vessel) is C_V=450 kJ/°C.

The latent heat of the steam may be approximated as λ_S=2300 kJ/kg (independent of the actual pressure), and the water's specific heat may be taken as $c_{p,w}$=4.18 kJ/(kg°C) (independent of the actual temperature).

The temperature in the vessel is measured by a mercury thermometer, its time constant is T_{Th} = 3 min.

Due to some failure in the steam line, 2 kg steam enters the vessel during negligible time.

Questions

a. What is the function $\vartheta_V(t)$ [°C]?

b. What is the function $\vartheta_{Th}(t)$ [°C]?

c. What is the highest temperature shown by the thermometer?

Solution

a. What is the function $\vartheta_V(t)$ [°C]?

The vessel is a first order element with the following gain and time constant:

$$A_V = \frac{\lambda_S}{W_w \cdot \rho_w \cdot c_{p,w}} = \frac{2300\,\frac{kJ}{kg}}{0.5\,\frac{m^3}{h}\cdot 1000\,\frac{kg}{m^3}\cdot 4.18\,\frac{kJ}{kg\cdot °C}} = 1.1\,\frac{°C}{kg/h}$$

$$T_V = \frac{C_V}{W_w \cdot \rho_w \cdot c_{p,w}} = \frac{450\,\frac{kJ}{°C}}{0.5\,\frac{m^3}{h}\cdot 1000\,\frac{kg}{m^3}\cdot 4.18\,\frac{kJ}{kg\cdot °C}} = 0.215h$$

The sudden entering extra steam is an impulse disturbance a=2 kg.

The response of the first order lag to the impulse is

$$\hat{\vartheta}_V(t) = \frac{a\cdot A_V}{T_V}\cdot e^{-\frac{t}{T_V}}$$

$$\vartheta_V(t) = \vartheta_V(0) + \frac{a\cdot A_V}{T_V}\cdot e^{-\frac{t}{T_V}} = 85°C + \frac{2kg\cdot 1.1\frac{°C}{kg/h}}{0.215h}\cdot e^{-\frac{t[h]}{0.215h}}$$

$$\vartheta_V(t) = 85°C + 10.23°C\cdot e^{-\frac{t[h]}{0.215h}} = 85°C + 10.23°C\cdot e^{-\frac{t[min]}{12.9\,min}}$$

(The impulse response is that the temperature suddenly jumps up to a maximum and then gradually decreases.) The maximum value is $\vartheta_{V,max} = 85°C + 10.23°C\cdot e^{-0} = 95.23°C$.

b. What is the function $\vartheta_{Th}(t)$ [°C]?

Both the vessel and the thermometer are first order lags. Gain of the thermometer is A_{Th}=1 [°C/°C], its time constant T_{Th} is given.

The series of two first order lags is a second order lag. Its gain is the product of the two first order gains. Since the time constants of the two first order lags are different, the damping factor ξ>1, thus the impulse response is

$$\hat{\vartheta}_{Th}(t) = a \cdot A_V \cdot A_{Th} \cdot \frac{1}{T_V - T_{Th}} \left[e^{-\frac{t}{T_V}} - e^{-\frac{t}{T_{Th}}} \right]$$

$$\vartheta_{Th}(t) = \vartheta_{Th}(0) = a \cdot A_V \cdot A_{Th} \cdot \frac{1}{T_V - T_{Th}} \left[e^{-\frac{t}{T_V}} - e^{-\frac{t}{T_{Th}}} \right]$$

In the initial steady state the thermometer shows the temperature of the vessel: $\vartheta_{Th}(0)$ = 85°C.

$$\vartheta_{Th}(t) = 85°C + 2kg \cdot 1.1 \frac{°C}{kg/h} \cdot 1 \cdot \frac{1}{0.215h - 0.05h} \left[e^{-\frac{t[h]}{0.215h}} - e^{-\frac{t[h]}{0.05h}} \right]$$

$$\vartheta_{Th}(t) = 85°C + 13.33°C \cdot \left[e^{-\frac{t[h]}{0.215h}} - e^{-\frac{t[h]}{0.05h}} \right] = 85°C + 13.33°C \cdot \left[e^{-\frac{t[min]}{12.9min}} - e^{-\frac{t[min]}{3min}} \right]$$

c. What is the highest temperature shown by the thermometer?

The impulse response of the second order lag has a maximum. This maximum is at the time where this function's derivative is zero. First we determine the place of the maximum:

$$\frac{d\vartheta_{Th}(t)}{dt} = 13.33°C \cdot \left[e^{-\frac{t[min]}{12.9min}} \cdot \left(-\frac{1}{12.9min} \right) - e^{-\frac{t[min]}{3min}} \cdot \left(-\frac{1}{3min} \right) \right] = 0$$

$$e^{-\frac{t[min]}{12.9min}} \cdot \left(-\frac{1}{12.9min} \right) - e^{-\frac{t[min]}{3min}} \cdot \left(-\frac{1}{3min} \right) = 0,$$

$$\frac{1}{12.9min} \cdot e^{-\frac{t[min]}{12.9min}} = \frac{1}{3min} \cdot e^{-\frac{t[min]}{3min}}$$

$$\ln \frac{3min}{12.9min} = \frac{t[min]}{12.9min} - \frac{t[min]}{3min} = \left(\frac{1}{12.9min} - \frac{1}{3min} \right) \cdot t[min]$$

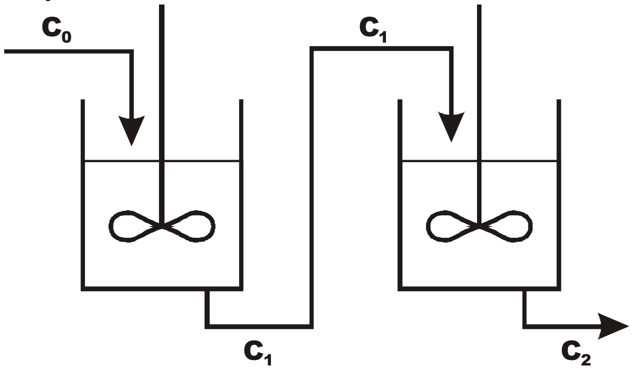
$$t_{max} = \frac{\ln \frac{3min}{12.9min}}{\frac{1}{12.9min} - \frac{1}{3min}} = 5.7 \text{ min}$$

Then the maximum can be calculated by substituting this time moment:

$$\vartheta_{Th,max}(t) = 85°C + 13.33°C \cdot \left[e^{-\frac{5.7[min]}{12.9min}} - e^{-\frac{5.7[min]}{3min}} \right] = 91.58°C$$

14.3 Problem 3: Capacities, 2nd order reaction

W=80 l/h solution is made in the first perfectly mixed tank of volume V_T=25 liter from (W=80 l/h) solvent and n(0)=0.5 kmol/h chemical 'Z'. The solution is warmed up and fed to a second, again perfectly mixed, isotherm tank reactor of volume V_R=50 liter, where chemical 'Z' reacts away in a chemical reaction of order 2.



In the initial steady state the concentration of 'Z' in the stream leaving the reactor is $c_2(0)$=1.5 kmol/m^3.
The flow rate of chemical 'Z' is suddenly lifted up from n(0)=0.5 kmol/h to n=0.65kmol/h.

Question

What is function $c_2(t)$, i.e. how the concentration of 'Z' in the stream leaving the reactor will change in time, and what will be its value at 25 min after the step disturbance?

Solution

Concentration c_0 of 'Z' in the feed stream in the steady state can be calculated directly:

$$c_0(0) = \frac{n(0)}{W_0} = \frac{0.5\frac{\text{kmol}}{\text{h}}}{0.08\frac{\text{m}^3}{\text{h}}} = 6.25\frac{\text{kmol}}{\text{m}^3}$$

We have a second order lag built up from two first order capacities. First we determine their parameters.

Solver tank

The solver tank does not do anything else than homogenizes the stream, thus its gain is

$$A_T = 1$$

Its time constant is

$$T_T = \frac{V_T}{W} = \frac{25\text{l}}{80\frac{\text{l}}{\text{h}}} = 0.313\text{h}$$

Reactor

The solver tank does not do anything else than homogenizes the stream, i.e.

$$c_1(0) = c_0(0) = 6.25\frac{\text{kmol}}{\text{m}^3}$$

Having this and the given $c_2(0)$, we can calculate the reaction rate from the material balance in the steady state:

$$W \cdot c_1(0) = W \cdot c_2(0) + V_R \cdot r(0)$$

where r is reaction rate.

$$r(0) = \frac{W}{V_R} \cdot [c_1(0) - c_2(0)] = \frac{0.08 \frac{m^3}{h}}{0.05 m^3} \cdot \left[6.25 \frac{kmol}{m^3} - 1.5 \frac{kmol}{m^3}\right] = 7.6 \frac{kmol}{m^3 \cdot h}$$

Now we can calculate the reaction rate coefficient k. We have a second order reaction, thus $r(0) = k \cdot [c_2(0)]^2$, and

$$k = \frac{r(0)}{[c_2(0)]^2} = \frac{7.6 \frac{kmol}{m^3 \cdot h}}{\left[1.5 \frac{kmol}{m^3}\right]^2} = 3.38 \frac{1}{h \cdot \frac{kmol}{m^3}}$$

In order to calculate the gain and the time constant of the reactor, we need the derivative in the working point:

$$\left(\frac{dr}{dc_2}\right)_0 = 2 \cdot k \cdot c_2(0) = 2 \cdot 3.38 \frac{1}{h \cdot \frac{kmol}{m^3}} \cdot 1.5 \frac{kmol}{m^3} = 10.13 \frac{1}{h}$$

$$A_R = \frac{W}{W + V_R \cdot \left(\frac{dr}{dc_2}\right)_0} = \frac{0.08 \frac{m^3}{h}}{0.08 \frac{m^3}{h} + 0.05 m^3 \cdot 10.13 \frac{1}{h}} = 0.136$$

$$T_R = \frac{V_R}{W + V_R \cdot \left(\frac{dr}{dc_2}\right)_0} = \frac{0.05 m^3}{0.08 \frac{m^3}{h} + 0.05 m^3 \cdot 10.13 \frac{1}{h}} = 0.085 h = 5.1 min$$

In order to calculate the response function, we have to calculate first the disturbance in c_0. As a result of increasing the chemical flow rate from n_0 to n, the new concentration is

$$c_0(\infty) = \frac{n}{W_0} = \frac{065 \frac{kmol}{h}}{0.08 \frac{m^3}{h}} = 8.125 \frac{kmol}{m^3}$$

Thus the step increase in c_0 is

$$a = c_0(\infty) - c_0(0) = 8.125 \frac{kmol}{m^3} - 6.25 \frac{kmol}{m^3} = 1.875 \frac{kmol}{m^3}$$

We have a pair of first order lags with different time constants, thus the damping factor $\xi > 1$, and the step response is

$$\hat{c}_2(i) = a \cdot A_T \cdot A_R \cdot \left[1 - \frac{1}{T_T - T_R}\left(T_T \cdot e^{-\frac{t}{T_T}} - T_R \cdot e^{-\frac{t}{T_R}}\right)\right]$$

$$c_2(t) = c_2(0) + a \cdot A_T \cdot A_R \cdot \left[1 - \frac{1}{T_T - T_R}\left(T_T \cdot e^{-\frac{t}{T_T}} - T_R \cdot e^{-\frac{t}{T_R}}\right)\right]$$

$$c_2(t) = 1.5\frac{kmol}{m^3} + 1.875\frac{kmol}{m^3/h} \cdot 1 \cdot 0.136 \cdot$$

$$\left[1 - \frac{1}{0.313h - 0.085h}\left(0.313h \cdot e^{-\frac{t[h]}{0.313h}} - 0.085h \cdot e^{-\frac{t[h]}{0.085h}}\right)\right]$$

$$c_2(t) = 1.5\frac{kmol}{m^3} + 0.255\frac{kmol}{m^3/h} \cdot \left[1 - 4.386\frac{1}{h}\left(0.313h \cdot e^{-\frac{t[h]}{0.313h}} - 0.085h \cdot e^{-\frac{t[h]}{0.085h}}\right)\right]$$

25 min is approximately 0.417 h, thus

$$c_2(0.417h) = 1.5\frac{kmol}{m^3} + 0.255\frac{kmol}{m^3/h} \cdot \left[1 - 4.386\frac{1}{h}\left(0.313h \cdot e^{-\frac{0.417h}{0.313h}} - 0.085h \cdot e^{-\frac{0.417h}{0.085h}}\right)\right]$$

$$c_2(25\,min) = c_2(0.417h) = 1.66\frac{kmol}{m^3}$$

14.4 Problem 4: Valve

50 m^3/h cooling water is needed in a heat exchanger, according to process design. With such a flow rate, the resistance against the flow in the heat exchanger and the pipeline is 2 bar. For controlling the flow rate, a control valve of exponential characteristic (with n=3) and throughput number $k_{v,max}$= 50 m^3/h is built in. The cooling water is circulated by a pump that provides with 4 bar constant pressure.

<u>Questions</u>

a. What is the valve position (how much is the valve open) at the following flow rates?
   - 30 m^3/h
   - 50 m^3/h
   - 60 m^3/h

b. What is the maximum possible flow rate?

<u>Solution</u>

a. <u>What is the valve position (how much is the valve open) at the following flow rates?</u>

The pressure drop is quadratic in the flow rate:

$$\Delta p_{pipe} = K \cdot W^2$$

$$K = \frac{\Delta p_{pipe}}{W^2} = \frac{2bar}{\left(50\frac{m^3}{h}\right)^2} = 8 \cdot 10^{-4} \frac{bar}{\left[{m^3}/{h}\right]^2}$$

Relative density of water: ρ_{rel} = 1.

<u>30 m^3/h</u>

Pressure drop on the heat exchanger and the pipeline at this flow rate:

$$\Delta p_{pipe} = K \cdot W^2 = 8 \cdot 10^{-4} \frac{bar}{\left[{m^3}/{h}\right]^2} \cdot \left(30\frac{m^3}{h}\right)^2 = 0.72bar$$

Pressure drop on the valve is the complementary:

$$\Delta p_{valve} = \Delta p_{total} - \Delta p_{pipe} = 4bar - 0.72bar = 3.28bar$$

$$\Delta p_{rel} = \frac{\Delta p_{valve}}{1bar} = 3.28$$

Maximum flow rate at this pressure drop on the valve:

$$W_{max} = k_{v,max} \cdot \sqrt{\frac{\Delta p_{rel}}{\rho_{rel}}} = 50\frac{m^3}{h} \cdot \sqrt{\frac{3.28}{1}} = 90.55\frac{m^3}{h}$$

The relation between W, W_{max}, and valve position (openness) h:

$$\frac{W}{W_{max}} = \frac{1}{e^n} \cdot e^{n \cdot h}$$

$$\frac{30\frac{m^3}{h}}{90.55\frac{m^3}{h}} = \frac{1}{e^3} \cdot e^{3 \cdot h}$$

$$h = 0.632$$

That is, at W=30 m^3/h the valve is open to 63.2 %.

<u>50 m^3/h</u>

Pressure drop on the heat exchanger and the pipeline at this flow rate:

$$\Delta p_{pipe} = K \cdot W^2 = 8 \cdot 10^{-4} \frac{bar}{\left[m^3/h \right]^2} \cdot \left(50\frac{m^3}{h} \right)^2 = 2bar$$

Pressure drop on the valve is the complementary:

$$\Delta p_{valve} = \Delta p_{total} - \Delta p_{pipe} = 4bar - 2bar = 2bar$$

$$\Delta p_{rel} = \frac{\Delta p_{valve}}{1bar} = 2$$

Maximum flow rate at this pressure drop on the valve:

$$W_{max} = k_{v,max} \cdot \sqrt{\frac{\Delta p_{rel}}{\rho_{rel}}} = 50\frac{m^3}{h} \cdot \sqrt{\frac{2}{1}} = 70.71\frac{m^3}{h}$$

The relation between W, W_{max}, and valve position (openness) h:

$$\frac{W}{W_{max}} = \frac{1}{e^n} \cdot e^{n \cdot h}$$

$$\frac{30\frac{m^3}{h}}{70.71\frac{m^3}{h}} = \frac{1}{e^3} \cdot e^{3 \cdot h}$$

$$h = 0.884$$

That is, at W=50 m^3/h the valve is open to 88.4 %.

<u>50 m^3/h</u>

Pressure drop on the heat exchanger and the pipeline at this flow rate:

$$\Delta p_{pipe} = K \cdot W^2 = 8 \cdot 10^{-4} \frac{bar}{\left[m^3/h \right]^2} \cdot \left(60\frac{m^3}{h} \right)^2 = 2.88bar$$

Pressure drop on the valve is the complementary:

$$\Delta p_{valve} = \Delta p_{total} - \Delta p_{pipe} = 4\text{bar} - 2.88\text{bar} = 1.12\text{bar}$$

$$\Delta p_{rel} = \frac{\Delta p_{valve}}{1\text{bar}} = 1.12$$

Maximum flow rate at this pressure drop on the valve:

$$W_{max} = k_{v,max} \cdot \sqrt{\frac{\Delta p_{rel}}{\rho_{rel}}} = 50\frac{m^3}{h} \cdot \sqrt{\frac{1.12}{1}} = 52.92\frac{m^3}{h}$$

This maximum flow rate is smaller than the assumed 60 m^3/h, and the actual flow rate can be smaller only. Therefore **60 m^3/h cannot be reached**.

b. <u>What is the maximum possible flow rate?</u>

The valve formula is

$$W_{max} = k_{v,max}\sqrt{\frac{\Delta p_{valve,rel}}{\rho_{rel}}} = k_{v,max}\sqrt{\frac{\frac{\Delta p_{total} - K \cdot W^2}{1\text{bar}}}{\rho_{rel}}}$$

The maximum possible flow rate is available at maximum opening, i.e. no characteristic is needed to calculate. In that case, however, W_{max} takes place at both sides of the formula:

$$W_{max} = k_{v,max}\sqrt{\frac{\frac{\Delta p_{total} - K \cdot W_{max}^2}{1\text{bar}}}{\rho_{rel}}}$$

$$W_{max}^2 = \frac{k_{v,max}^2 \cdot \left(\Delta p_{total} - K \cdot W_{max}^2\right)}{1\text{bar} \cdot \rho_{rel}}$$

Thus the maximum possible flow rate is

$$W_{max}^* = \sqrt{\frac{\frac{\Delta p_{total}}{1\text{bar}}}{\frac{\rho_{rel}}{k_{v,max}^2} + \frac{K}{1\text{bar}}}} = \sqrt{\frac{\frac{4\text{bar}}{1\text{bar}}}{\frac{1}{\left[50\frac{m^3}{h}\right]^2} + \frac{8 \cdot 10^{-4}\frac{\text{bar}}{\left[m^3/h\right]^2}}{1\text{bar}}}} = 57.74\frac{m^3}{h}$$

14.5 Problem 5: Valve

Water flows in a long pipe, driven by constant 2 bar. An exponential characteristic (with n=3) and throughput number $k_{v,max}$= 50 m^3/h control valve is built in the pipe. At 40% opening the flow rate is 8.3 m3/h.

Questions

a. What is the valve position at 10 m^3/h?
b. What is the valve position at 12 m^3/h?

Solution

For answering, we need to know how the pressure drop over the pipe depends on the flow rate. As the only known flow rate belongs to a partial opening, we have to take into account the characteristic as well.

For this aim, we first calculate the maximum flow rate W_{max} at 40% opening (h=0.4) when the flow rate W=8.3 m3/h:

$$\frac{W}{W_{max}} = \frac{1}{e^n} \cdot e^{n \cdot h}$$

$$\frac{8.3\frac{m^3}{h}}{W_{max}} = \frac{1}{e^3} \cdot e^{3 \cdot 0.4}$$

$$W_{max} = 50.21\frac{m^3}{h}$$

Now we can calculate the pressure drop on the valve from the valve formula:

$$W_{max} = k_{v,max} \cdot \sqrt{\frac{\Delta p_{rel}}{\rho_{rel}}}$$

$$50.21\frac{m^3}{h} = 50\frac{m^3}{h} \cdot \sqrt{\frac{\Delta p_{rel}}{1}}$$

$$\Delta p_{rel} \approx 1$$

$$\Delta p_{valve} = \Delta p_{rel} \cdot 1 bar = 1 bar$$

We can now calculate the pressure drop over the pipe, and then characterize the pipe resistance:

$$\Delta p_{pipe} = \Delta p_{total} - \Delta p_{valve} = 2bar - 1bar = 1bar$$

$$\Delta p_{pipe} = K \cdot W^2$$

$$B = \frac{\Delta p_{pipe}}{W^2} = \frac{1bar}{\left(8.3\frac{m^3}{h}\right)^2} = 1.45 \cdot 10^{-2}\,\frac{bar}{\left[m^3/h\right]^2}$$

a. <u>What is the valve position at 10 m^3/h?</u>

$$\Delta p_{pipe} = K \cdot W^2 = 1.45 \cdot 10^{-2} \frac{bar}{\left[m^3/h\right]^2} \cdot \left(10 \frac{m^3}{h}\right)^2 = 1.45 bar$$

$$\Delta p_{valve} = \Delta p_{total} - \Delta p_{pipe} = 2 bar - 1.45 bar = 0.55 bar$$

$$\Delta p_{rel} = \frac{\Delta p_{valve}}{1 bar} = 0.55$$

$$W_{max} = k_{v,max} \cdot \sqrt{\frac{\Delta p_{rel}}{\rho_{rel}}} = 50 \frac{m^3}{h} \cdot \sqrt{\frac{0.55}{1}} = 37.1 \frac{m^3}{h}$$

$$\frac{W}{W_{max}} = \frac{1}{e^n} \cdot e^{n \cdot h}$$

$$\frac{10 \frac{m^3}{h}}{37.1 \frac{m^3}{h}} = \frac{1}{e^3} \cdot e^{3 \cdot h}$$

$$h = 0.563$$

The valve is open to 56.3%.

b. <u>What is the valve position at 12 m^3/h?</u>

$$\Delta p_{pipe} = B \cdot W^2 = 1.45 \cdot 10^{-2} \frac{bar}{\left[m^3/h\right]^2} \cdot \left(12 \frac{m^3}{h}\right)^2 = 2.09 bar$$

This, however, is larger than the total 2 bar, thus **12 m3/h cannot be achived with any large valve. This is over the limit of the pipeline and pump system itself.**

The absolute limit, considering an infinite large valve, is as follows:

$$\Delta p_{pipe} = K \cdot W^2$$

$$W = \sqrt{\frac{\Delta p_{pipe}}{K}}$$

$$W = \sqrt{\frac{2 bar}{1.45 \cdot 10^{-2} \frac{bar}{\left[m^3/h\right]^2}}} = 11.744 \frac{m^3}{h}$$

14.6 Problem 6: Level control loop

Level is controlled in a vertical cylindrical water tank of diameter D=3 m. Domain of the transmitter is 0.6 m wide. The transmitter is of linear characteristic. The outlet stream is transported by a pump that provides with 0.5 bar pressure difference independently of the flow rate. A linear working characteristic control valve with throughput number $k_{v,max}$= 25 m^3/h is built in the outlet pipe. The outlet pipe is wide, its resistance may be neglected.
A P-controller is used with gain A_C=20.

In the initial steady state the level is 1.25 m, the water flow rate is 10 m^3/h.

Questions

a. In what range of the flow rate can the level be kept constant?

b. In what range will the level change?

Solution

a. In what range of the flow rate can the level be kept constant?
The lowest flow rate is, naturally, 0 m^3/h. In such a case the valve closes full, and the level remains.
It is just the valve that limits the flow rate from above, because the pipe itself has no resistance. The limit is the maximum flow rate the valve can let through.

$$\Delta p_{rel} = \frac{\Delta p_{valve}}{1\text{bar}} = \frac{\Delta p_{pump}}{1\text{bar}} = \frac{0.5\text{bar}}{1\text{bar}} = 0.5$$

$$W_{max} = k_{v,max} \cdot \sqrt{\frac{\Delta p_{rel}}{\rho_{rel}}} = 25\frac{m^3}{h} \cdot \sqrt{\frac{0.5}{1}} = 17.68\frac{m^3}{h}$$

Thus, W∈[0 m^3/h to 17.68 m^3/h].

b. In what range will the level change?
For answering this question, we need the transfer function from the (inlet) flow rate (disturbance) to the level (controlled variable) taking into account the effects of the control loop. For this, we need the transfer function of all the elements in the loop: the process, the transmitter, the controller, and the actuator.

$$G^*(s) = \frac{L(s)}{W(s)} = \frac{G_{Pr}(s)}{1 + G_{Pr}(s) \cdot G_{Tr}(s) \cdot G_C(s) \cdot G_{Ac}(s)}$$

Process

The process is a tank level with forced outflow, thus an integrating element:

$$G_{Pr}(s) = \frac{A_{Pr}}{s}$$

Its gain is:

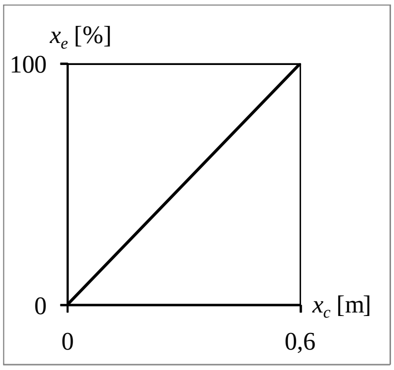
$$A_{Pr} = \frac{1}{B} = \frac{1}{\frac{D^2\pi}{4}} = \frac{1}{\frac{(3m)^2\pi}{4}} = 0.14\frac{1}{m^2}$$

<u>Transmitter</u>

We have a linear transmitter, its model is a proportional element:

$$G_{Tr}(s) = A_{Tr}$$

The transmitter measures the level L in a range of L_0 to L_1 in a way that L_0–L_1 is 0.6 m. Let x_c denote the location in this range, and let x_e denote the control signal between 0 and 100%. Then the characteristic is a straight line according to the figure below.



The transmitter gain is the slope of this straight line:

$$A_{Tr} = \frac{100\% - 0\%}{0.6m - 0m} = 166.67\frac{\%}{m}$$

<u>Controller</u>

We have a P controller with given gain:

$$G_C(s) = A_C = 20$$

<u>Actuator</u>

We have a linear valve (linear working characteristic), its model is a proportional element:

$$G_{Ac}(s) = A_{Ac}$$

The flow rate changes 0 m^3/h to 17.68 m^3/h as a result of the command changing 0 to 100%, thus the gain is

$$A_{Ac} = \frac{17.68\frac{m^3}{h} - 0\frac{m^3}{h}}{100\% - 0\%} = 0.177\frac{m^3/h}{\%}$$

<u>Resultant transfer function</u>

$$G^*(s) = \frac{\frac{A_{Pr}}{s}}{1 + \frac{A_{Pr}}{s} \cdot A_{Tr} \cdot A_C \cdot A_{Ac}}$$

$$G^*(s) = \frac{\frac{1}{A_{Tr} \cdot A_C \cdot A_{Ac}}}{\frac{1}{A_{Pr} \cdot A_{Tr} \cdot A_C \cdot A_{Ac}} \cdot s + 1} = \frac{A^*}{T^* \cdot s + 1}$$

This is a first order lag with parameters:

$$A^* = \frac{1}{A_{Tr} \cdot A_C \cdot A_{Ac}} = \frac{1}{166.67\,\frac{\%}{m} \cdot 20 \cdot 0.177\,\frac{m^3/h}{\%}} = 1.7 \cdot 10^{-3}\,\frac{m}{m^3/h}$$

$$T^* = \frac{1}{A_{Pr} \cdot A_{Tr} \cdot A_C \cdot A_{Ac}} = \frac{1}{0.14\,\frac{1}{m^2} \cdot 166.67\,\frac{\%}{m} \cdot 20 \cdot 0.177\,\frac{m^3/h}{\%}} = 1.21 \cdot 10^{-2} h = 0.73 \min$$

<u>Step disturbance from steady state to minimum flow rate</u>

$$a = W_{new} - W(0) = 0\,\frac{m^3}{h} - 10\,\frac{m^3}{h} = -10\,\frac{m^3}{h}$$

$$\hat{L}(\infty) = a \cdot A^* = -10\,\frac{m^3}{h} \cdot 1.7 \cdot 10^{-3}\,\frac{m}{m^3/h} = -0.017m$$

$$L(\infty) = L(0) + \hat{L}(\infty) = 1.25m - 0.017m = 1.233m$$

<u>Step disturbance from steady state to maximum flow rate</u>

$$a = W_{new} - W(0) = 17.68\,\frac{m^3}{h} - 10\,\frac{m^3}{h} = 7.68\,\frac{m^3}{h}$$

$$\hat{L}(\infty) = a \cdot A^* = 7.68\,\frac{m^3}{h} \cdot 1.7 \cdot 10^{-3}\,\frac{m}{m^3/h} = 0.013m$$

$$L(\infty) = L(0) + \hat{L}(\infty) = 1.25m + 0.013m = 1.263m$$

Thus the level will change from 1.233 m to 1.263 m, in a range of 3 cm only.

14.7 Problem 7: Level control loop

A 2 m high vertical water tank of horizontal cross section area B=4.5 m^2 is used for balancing flow rate fluctuations. The outlet stream is transported by a pump that provides some pressure difference independently of the flow rate.
Additional elements of the control loop are:

- Transmitter, linear, domain length 3 m
- P controller, gain A_C=1.2
- Valve, exponential basic characteristic, approximately linear working characteristic in the command domain 15% to 58%, letting through 3 – 15 m^3/h in this domain.

In a steady state the flow rate is 3 m^3/h and the level is 0.25 m.
The level should be constrained inside the interval 0.25 m to 1.75 m.

Questions

a. What minimum and maximum inlet flow rate may be permitted?
b. What can be the minimum and the maximum outlet flow rate?
c. What is the time constant of the loop?

Solution

a. What minimum and maximum inlet flow rate may be permitted?
Only such flow rate is permitted at which the level remains in the specified interval 0.25 m to 1.75 m. Starting from a known steady state we are looking for a step disturbance that pushes the level to the limit.
For this aim first the gain of the loop from the inlet flow rate to the level is determined.

$$G^*(s) = \frac{L(s)}{W_{in}(s)} = \frac{G_{Pr}(s)}{1 + G_{Pr}(s) \cdot G_{Tr}(s) \cdot G_C(s) \cdot G_{Ac}(s)}$$

We need the transfer function of all the elements in the loop: the process, the transmitter, the controller, and the actuator.
Process
The process is a tank level with forced outflow, thus an integrating element:

$$G_{Pr}(s) = \frac{A_{Pr}}{s}$$

Its gain is:

$$A_{Pr} = \frac{1}{B} = \frac{1}{4.5m^2} = 0.22\frac{1}{m^2}$$

Transmitter
We have a linear transmitter, its model is a proportional element:

$$G_{Tr}(s) = A_{Tr}$$

$$A_{Tr} = \frac{100\% - 0\%}{3m - 0m} = 33.33\frac{\%}{m}$$

Controller

We have a P controller with given gain:

$$G_C(s) = A_C = 1.2$$

Actuator

The valve is of linear working characteristic, its model is a proportional element:

$$G_{Ac}(s) = A_{Ac}$$

$$A_{Ac} = \frac{15\frac{m^3}{h} - 3\frac{m^3}{h}}{85\% - 15\%} = 0.171\frac{m^3/h}{\%}$$

Resultant transfer function

$$G^*(s) = \frac{\frac{A_{Pr}}{s}}{1 + \frac{A_{Pr}}{s} \cdot A_{Tr} \cdot A_C \cdot A_{Ac}}$$

$$G^*(s) = \frac{\frac{1}{A_{Tr} \cdot A_C \cdot A_{Ac}}}{\frac{1}{A_{Pr} \cdot A_{Tr} \cdot A_C \cdot A_{Ac}} \cdot s + 1} = \frac{A^*}{T^* \cdot s + 1}$$

$$A^* = \frac{1}{A_{Tr} \cdot A_C \cdot A_{Ac}} = \frac{1}{33.33\frac{\%}{m} \cdot 1.2 \cdot 0.171\frac{m^3/h}{\%}} = 0.146\frac{m}{m^3/h}$$

Step disturbance from steady state

Step down: The steady state is just at the lower limit of both the level and the flow rate working domain of the valve, therefore the lower limit is 3 m^3/h (no step down).

Step up:

$$\hat{L} = L_{max} - L(0) = 1.75\text{ m} - 0.25\text{ m} = a \cdot A^* = a \cdot 0.146\frac{m}{m^3/h}$$

$$a = \frac{1.75\text{m} - 0.25\text{ m}}{0.146\frac{m}{m^3/h}} = 10.27\, m^3/h$$

$$W_{max} = W(0) + a = 3\frac{m^3}{h} + 10.274\frac{m^3}{h} = 13.27\frac{m^3}{h}$$

The inlet flow rate domain is 3 m^3/h to 13.27 m^3/h.

b. What can be the minimum and the maximum outlet flow rate?

Since in steady state the inlet and the outlet are equal, this is the same as for the inlet: 3 m^3/h to 13.27 m^3/h.

c. <u>What is the time constant of the loop?</u>

$$T^* = \frac{1}{A_{Pr} \cdot A_{Tr} \cdot A_C \cdot A_{Ac}} = \frac{1}{0.22 \frac{1}{m^2} \cdot 33.33 \frac{\%}{m} \cdot 1.2 \cdot 0.171 \frac{m^3/h}{\%}} = 0.665h = 39.9min$$

14.8 Problem 8: Level control loop

A vertical water tank of diameter D=2.5 m is used for balancing flow rate fluctuations. The outlet stream is transported by a pump that provides 2 bar pressure difference independently of the flow rate. Resistance of the pipe may be neglected. The control valve is of linear working characteristic, $k_{v,max}$ = 12 m^3/h. The transmitter domain is 2.5 m wide, its center (50 % control signal) is set to level 1.5 m. The set point signal is such that at level 0.5 m the valve is open to 10 %.
Our target is to keep the level in 0.5 m to 2.5 m in a way that the valve is open in a range 10 % to 90 %.

<u>Questions</u>

a. What should be the controller gain?
b. What is the flow rate if the level is 1.5 m in steady state?
c. From the steady state of question b, another input pipe is suddenly open, and a new stream starts with either (c1) 5 m^3/h or (c2) 10 m^3/h. How does the level change in time in both cases?

<u>Solution</u>

a. <u>What should be the controller gain?</u>
For this aim first the gain of the loop from the inlet flow rate to the level is to be determined.

$$G^*(s) = \frac{L(s)}{W_{in}(s)} = \frac{G_{Pr}(s)}{1 + G_{Pr}(s) \cdot G_{Tr}(s) \cdot G_C(s) \cdot G_{Ac}(s)}$$

We need the transfer function of all the elements in the loop: the process, the transmitter, the controller, and the actuator.

<u>Process</u>

The process is a tank level with forced outflow, thus an integrating element:

$$G_{Pr}(s) = \frac{A_{Pr}}{s}$$

Its gain is:

$$A_{Pr} = \frac{1}{\frac{D^2\pi}{4}} = \frac{1}{\frac{(2.5m)^2\pi}{4}} = 0.2\frac{1}{m^2}$$

<u>Transmitter</u>

We have a linear transmitter, its model is a proportional element:

$$G_{Tr}(s) = A_{Tr}$$

$$A_{Tr} = \frac{100\% - 0\%}{2.5m - 0m} = 40\frac{\%}{m}$$

Controller

We have a P controller with unknown gain:

$$G_C(s) = A_C$$

Actuator

The valve is of linear working characteristic, its model is a proportional element:

$$G_{Ac}(s) = A_{Ac}$$

The maximum flow rate over the valve:

$$\Delta p_{rel} = \frac{\Delta p_{valve}}{1bar} = \frac{\Delta p_{pump}}{1bar} = \frac{2bar}{1bar} = 2$$

$$W_{max} = k_{v,max} \cdot \sqrt{\frac{\Delta p_{rel}}{\rho_{rel}}} = 12\frac{m^3}{h} \cdot \sqrt{\frac{2}{1}} = 17\frac{m^3}{h}$$

Thus the gain is:

$$A_{Ac} = \frac{17\frac{m^3}{h} - 0\frac{m^3}{h}}{100\% - 0\%} = 0.17\frac{m^3/h}{\%}$$

Resultant transfer function

$$G^*(s) = \frac{\frac{A_{Pr}}{s}}{1 + \frac{A_{Pr}}{s} \cdot A_{Tr} \cdot A_C \cdot A_{Ac}}$$

$$G^*(s) = \frac{\frac{1}{A_{Tr} \cdot A_C \cdot A_{Ac}}}{\frac{1}{A_{Pr} \cdot A_{Tr} \cdot A_C \cdot A_{Ac}} \cdot s + 1} = \frac{A^*}{T^* \cdot s + 1}$$

$$A^* = \frac{1}{A_{Tr} \cdot A_C \cdot A_{Ac}} = \frac{1}{40\frac{\%}{m} \cdot A_C \cdot 0.17\frac{m^3/h}{\%}} = \frac{0.147}{A_C}\frac{m}{m^3/h}$$

$$T^* = \frac{1}{A_{Pr} \cdot A_{Tr} \cdot A_C \cdot A_{Ac}} = \frac{1}{0.2\frac{1}{m^2} \cdot 40\frac{\%}{m} \cdot A_C \cdot 0.17\frac{m^3/h}{\%}} = \frac{0.735}{A_C}h$$

Controller gain

If the valve position steps up from 10 % to 90 % then the level should step up from 0.5 m to 2.5 m.

The flow rate at 0.5 m:

$$W(0) = 0.1 \cdot W_{max} = 0.1 \cdot 17 \frac{m^3}{h} = 1.7 \frac{m^3}{h}$$

The flow rate at 2.5 m:

$$W_{new} = 0.9 \cdot W_{max} = 0.9 \cdot 17 \frac{m^3}{h} = 15.3 \frac{m^3}{h}$$

The step disturbance is:

$$a = W_{new} - W(0) = 15.3 \frac{m^3}{h} - 1.7 \frac{m^3}{h} = 13.6 \frac{m^3}{h}$$

The level should answer by changing from 0.5 m to 2.5 m:

$$\hat{L} = L(\infty) - L(0) = 2.5m - 0.5m = 2m$$

The change is, by definition, the step multiplied by the gain:

$$\hat{L} = a \cdot A^*$$

$$2m = 13.6 \frac{m^3}{h} \cdot \frac{0.147}{A_C} \frac{m}{m^3/h}$$

$$A_C = \frac{2m}{13.6 \frac{m^3}{h} \cdot 0.147 \frac{m}{m^3/h}} = 1$$

b. <u>What is the flow rate if the level is 1.5 m in steady state?</u>
Substitute the controller gain to get the loop gain:

$$A^* = \frac{0.147}{A_C} \frac{m}{m^3/h} = \frac{0.147}{1} \frac{m}{m^3/h} = 0.147 \frac{m}{m^3/h}$$

We know that at level 0.5 m the valve is open to 10 %. We have already calculated the flow rate at this opening: 1.7 m^3/h. From such a steady state a step disturbance arrived and the system responded by increasing the level to 1.5 m. First we calculate how much the disturbance was, and then we can calculate the flow rate.

$$\hat{L} = L(\infty) - L(0) = 1.5m - 0.5m = 1m$$

$$\hat{L} = a \cdot A^*$$

$$1m = a \cdot 0.147 \frac{m}{m^3/h}$$

$$a = \frac{1m}{0.147 \frac{m}{m^3/h}} = 6.8 \frac{m^3}{h}$$

$$W_{new} = W(0) + a = 1.7 \frac{m^3}{h} + 6.8 \frac{m^3}{h} = 8.5 \frac{m^3}{h}$$

c. From the steady state of question b, another input pipe is suddenly open, and a new stream starts with (c1) 5 m^3/h, (c2) 10 m^3/h. How does the level change in time in both cases?

The transfer function derived in part a is valid from W_{in} to L only. Here we need the transfer function from W_{in} to W_{out}:

$$G^*(s) = \frac{W_{out}(s)}{W_{in}(s)} = \frac{G_{Pr}(s)\cdot G_{Tr}(s)\cdot G_C(s)\cdot G_{Ac}(s)}{1+G_{Pr}(s)\cdot G_{Tr}(s)\cdot G_C(s)\cdot G_{Ac}(s)} = \frac{\frac{A_{Pr}}{s}\cdot A_{Tr}\cdot A_C\cdot A_{Ac}}{1+\frac{A_{Pr}}{s}\cdot A_{Tr}\cdot A_C\cdot A_{Ac}}$$

$$G^*(s) = \frac{1}{\frac{1}{A_{Pr}\cdot\cdot A_{Tr}\cdot A_C\cdot A_{Ac}}\cdot s+1} = \frac{A^\#}{T^\#\cdot s+1}$$

$$A^\# = 1$$

$$T^\# = \frac{1}{A_{Pr}\cdot\cdot A_{Tr}\cdot A_C\cdot A_{Ac}} = \frac{1}{0.2\frac{1}{m^2}\cdot 40\frac{\%}{m}\cdot 1\cdot 0.17\frac{m^3/h}{\%}} = 0.735h$$

We have step disturbance. The step response of this first order lag:

$$\hat{W}_{out}(t) = a\cdot A^\#\cdot\left[1-e^{-\frac{t}{T^\#}}\right]$$

$$W_{out}(t) = W_{out}(0)+\hat{W}_{out}(t) = 8.5\frac{m^3}{h}+a\cdot 1\cdot\left[1-e^{-\frac{t[h]}{0.735h}}\right]$$

c1: a=5 m^3/h

$$W_{out}(t) = 8.5\frac{m^3}{h}+5\frac{m^3}{h}\cdot\left[1-e^{-\frac{t[h]}{0.735h}}\right]$$

The final value is:

$$W_{out}(\infty) = 8.5\frac{m^3}{h}+5\frac{m^3}{h}\cdot 1 = 13.5\frac{m^3}{h}$$

c2: a=10 m^3/h

$$W_{out}(t) = 8.5\frac{m^3}{h}+10\frac{m^3}{h}\cdot\left[1-e^{-\frac{t[h]}{0.735h}}\right]$$

The final value would be:

$$W_{out}(\infty) = 8.5\frac{m^3}{h}+10\frac{m^3}{h}\cdot 1 = 18.5\frac{m^3}{h}$$

but the maximum flow rate is 17 m^3/h, and **new steady state cannot form**. After W_{out} reaches 17 m^3/h, the level increases linearly, and finally the water overflows.

14.9 Problem 9: Frequency function and tuning

The transfer function of the controlled section in a feed-back loop is

$$G(s) = \frac{0.6 \cdot e^{-(3\,\mathrm{min})\cdot s}}{[(12\,\mathrm{min}) \cdot s + 1]^3}$$

Tasks

a. Compute the frequency function of the controlled section and visualize it in a Bode plot!

b. Compute the tuning with PI controller according to the Ziegler-Nichols table!

c. Compute the frequency function of the open loop (with the tuning in task b) at frequency $\omega = 0.1$ 1/min!

Solution

a. Compute the frequency function of the controlled section and visualize it in a Bode plot!

The frequency function can be considered as a series of a dead time element with gain A_D and dead time T_D, and three identical first order elements with gain A and time constant T:

$$G(s) = A_D \cdot e^{-T_D \cdot s} \cdot \left(\frac{A}{T \cdot s + 1}\right)^3 = \left(A_D \cdot A^3\right) \cdot e^{-T_D \cdot s} \frac{1}{(T \cdot s + 1)^3}$$

where T_D=3 min and T=12 min. Decomposition of 0.6 to $A_D \cdot A^3$ can be done in several ways. For simplicity let us choose A=1 and A_D=0.6.

Properties of these element can be calculated independently, then the gains must be multiplied and the phase shifts added together to get the function of the whole section. Denote the dead time by index 1, and the third order element obtained by combining the three first order elements by index 2.

$$|G(i \cdot \omega)|_1 = |g(\omega)|_1 = A_D = 0.6$$

$$\varphi_1 = -57.3° \cdot T_D \cdot \omega = -57.3° \cdot (3\,\mathrm{min}) \cdot \omega = -(171.9°\,\mathrm{min}) \cdot \omega$$

(because 1 rad ≈ 57.3°)

$$|g(\omega)|_2 = \frac{1}{\left(\sqrt{T^2 \cdot \omega^2 + 1}\right)^3} = \frac{1}{\left(\sqrt{(12\,\mathrm{min})^2 \cdot \omega^2 + 1}\right)^3} = \frac{1}{\left(\sqrt{(144\,\mathrm{min})^2 \cdot \omega^2 + 1}\right)^3}$$

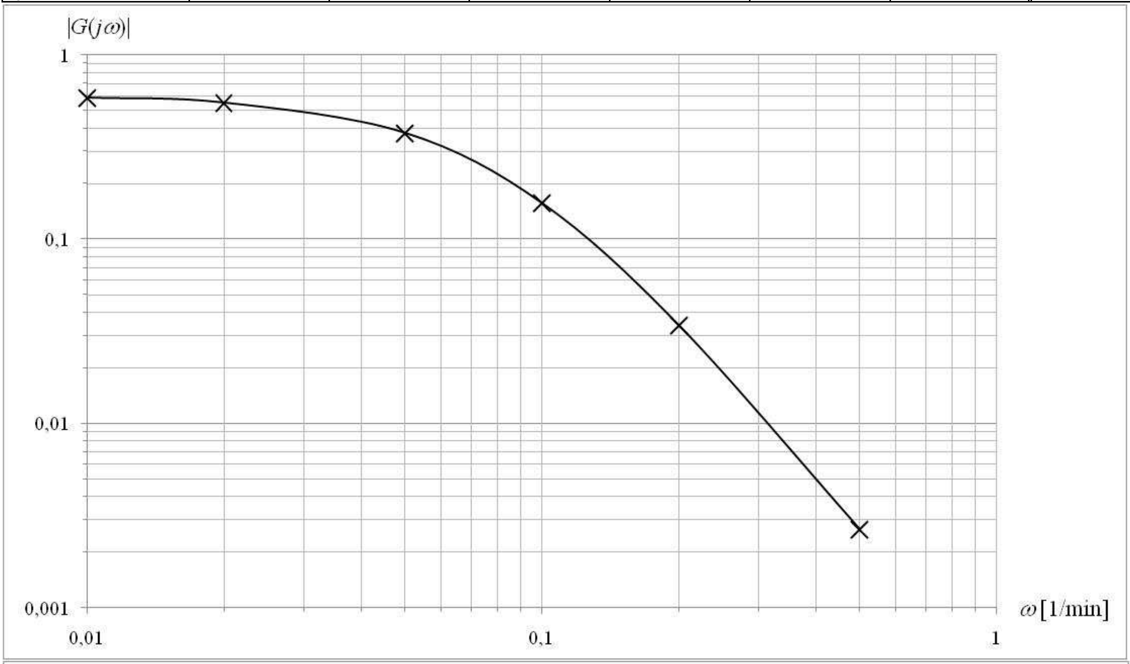
$$\varphi_2 = 3 \cdot \mathrm{arctan}(-T \cdot \omega) = 3 \cdot \mathrm{arctan}(-(12\,\mathrm{min}) \cdot \omega)$$

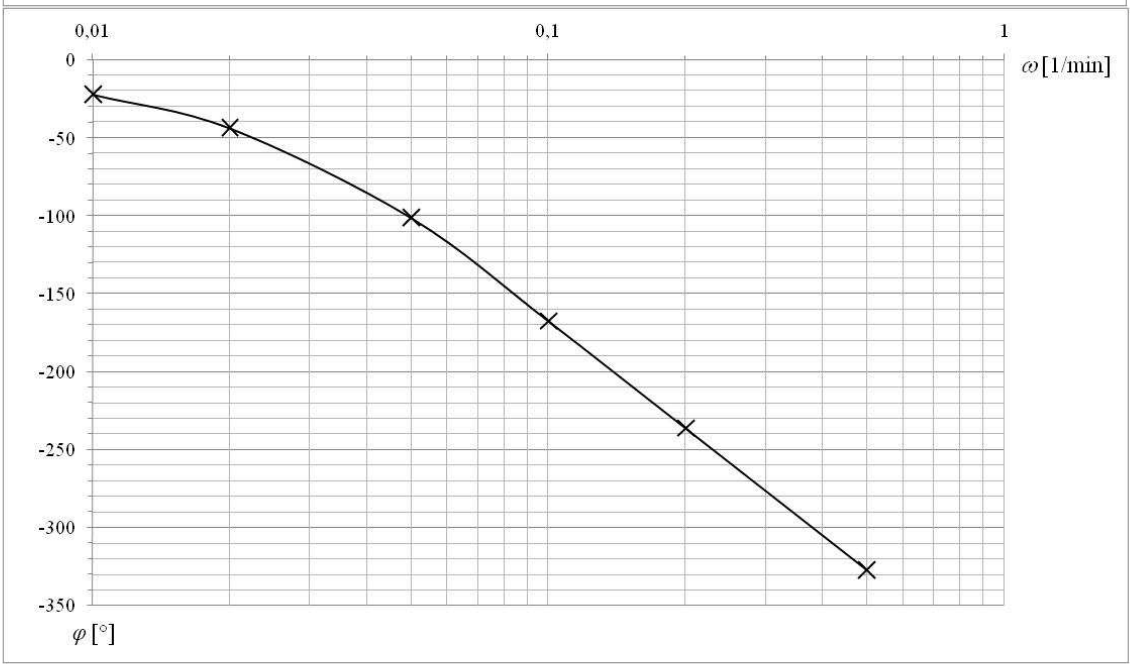
$$|G(i \cdot \omega)| = |g(\omega)| = |g(\omega)|_1 \cdot |g(\omega)|_2$$

$$\varphi = \varphi_1 + \varphi_2$$

The calculations are shown in a table below. (The last column belongs to Task b.)

ω [1/min]	0.01	0.02	0.05	0.1	0.2	0.5	0.11
$\lvert G(i\cdot\omega)\rvert_1$ [–]	0.6	0.6	0.6	0.6	0.6	0.6	0.6
φ_1 [°]	-1.72	-3.44	-8.60	-17.19	-34.38	-85.95	-18.91
$\lvert G(i\cdot\omega)\rvert_2$ [–]	0.979	0.919	0.631	0.262	0.057	$4.44\cdot10^{-3}$	0.220
φ_2 [°]	-20.53	-40.49	-92.89	-150.58	-202.14	-241.61	-158.56
$\lvert G(i\cdot\omega)\rvert$ [–]	0.587	0.551	0.379	0.157	0.034	$2.66\cdot10^{-3}$	0.132
φ [°]	-22.25	-43.93	-101.49	-167.77	-236.52	-327.56	-177.47





b. Compute the tuning with PI controller according to the Ziegler-Nichols table!

The critical data belong to the frequency at which the phase shift is –180°. Reading the Bode plot we find that this is at ω_0 = 0.11 1/min. In that point the gain of the controlled section is 0.132. The critical gain and the critical period are then

$$A_{C,crit} = \frac{1}{|g(\omega)|_{crit}} = \frac{1}{0.132} = 7.58$$

$$T_{crit} = \frac{2\pi}{\omega_0} = \frac{2\pi}{0.11\frac{1}{\min}} = 57.12\,\min$$

According to the Ziegler-Nichols table, the suggested tuning is:

$$A_C = 0.45 \cdot A_{C,crit} = 0.45 \cdot 7.58 = 3.41$$

$$I = \frac{T_{crit}}{1.2} = \frac{57.12\,\min}{1.2} = 47.6\,\min$$

c. Compute the frequency function of the open loop (with the tuning in task b) at frequency ω = 0.1 1/min!

Amplitude ratio of the section at ω = 0.1 1/min read from the table above:

$$|g(\omega)|_{\omega=0.1\frac{1}{\min}} = 0.157$$

Amplitude ratio of the controller at ω = 0.1 1/min

$$|g(\omega)|_{PI} = A_C \cdot \sqrt{1+\frac{1}{\omega^2 \cdot I^2}} = 3.41 \cdot \sqrt{1+\frac{1}{\left(0.1\frac{1}{\min}\right)^2 \cdot (47.6\,\min)^2}} = 3.48$$

Open loop amplitude ratio is their product:

$$|g^*(\omega)|_{\omega=0.1\frac{1}{\min}} = 0.157 \cdot 3.48 = 0.55$$

Phase shift of the section at ω = 0.1 1/min read from the table above:

$$\varphi\left(\omega = 0.1\frac{1}{\min}\right) = -167.77°$$

Phase shift of the controller at ω = 0.1 1/min:

$$\varphi_{PI} = \arctan\left(-\frac{1}{\omega \cdot I}\right) = \arctan\left(-\frac{1}{0.1\frac{1}{\min} \cdot 47.6\,\min}\right) = -11.85°$$

Open loop phase shift is their sum:

$$\varphi^*\left(\omega = 0.1\frac{1}{\min}\right) = -167.77° - 11.85° = -179.62°$$

14.10 Problem 10: Frequency function and tuning

The transfer function of the controlled section in a feed-back loop is

$$G(s) = e^{-(6\,\text{min})\cdot s} \frac{([15\,\text{min}]\cdot s + 0.75)}{([20\,\text{min}]\cdot s + 1)^3 \cdot ([30\,\text{min}\cdot s] + 1.5)}$$

<u>Tasks</u>

a. Compute the frequency function of the controlled section and visualize it in a Bode plot!

b. Compute the tuning with P controller according to the Ziegler-Nichols table!

c. Compute the phase margin of the closed loop!

d. Compute the gain margin of the closed loop!

<u>Solution</u>

a. <u>Compute the frequency function of the controlled section and visualize it in a Bode plot!</u>
First we simplify the expression:

$$G(s) = e^{-(6\,\text{min})\cdot s} \frac{0.75 \cdot ([20\,\text{min}]\cdot s + 1)}{([20\,\text{min}]\cdot s + 1)^3 \cdot 1.5 \cdot ([20\,\text{min}]\cdot s + 1)}$$

$$G(s) = e^{-(6\,\text{min})\cdot s} \frac{0.5}{([20\,\text{min}]\cdot s + 1)^3}$$

From here on, we can calculate in the same way as in Problem 9 above. The table is:

ω [1/min]	0,01	0,02	0,05	0,1	0,2	0,5	0,065	0,04
$\lvert g(\omega)\rvert_1$ [–]	1	1	1	1	1	1	1	1
φ_1 [°]	-3,44	-6,88	-17,19	-34,38	-68,76	-171,90	-22,35	-13,75
$\lvert g(\omega)\rvert_2$ [–]	0,471	0,400	0,177	0,045	$7{,}13\cdot10^{-3}$	$4{,}93\cdot10^{-4}$	0,113	0,238
φ_2 [°]	-33,93	-65,40	-135,00	-190,30	-227,89	-252,87	-157,29	-115,98
$\lvert g(\omega)\rvert$ [–]	0,471	0,400	0,177	0,045	$7{,}13\cdot10^{-3}$	$4{,}93\cdot10^{-4}$	0,113	0,238
φ [°]	-37,37	-72,28	-152,19	-224,68	-296,65	-424,77	-179,64	-129,73

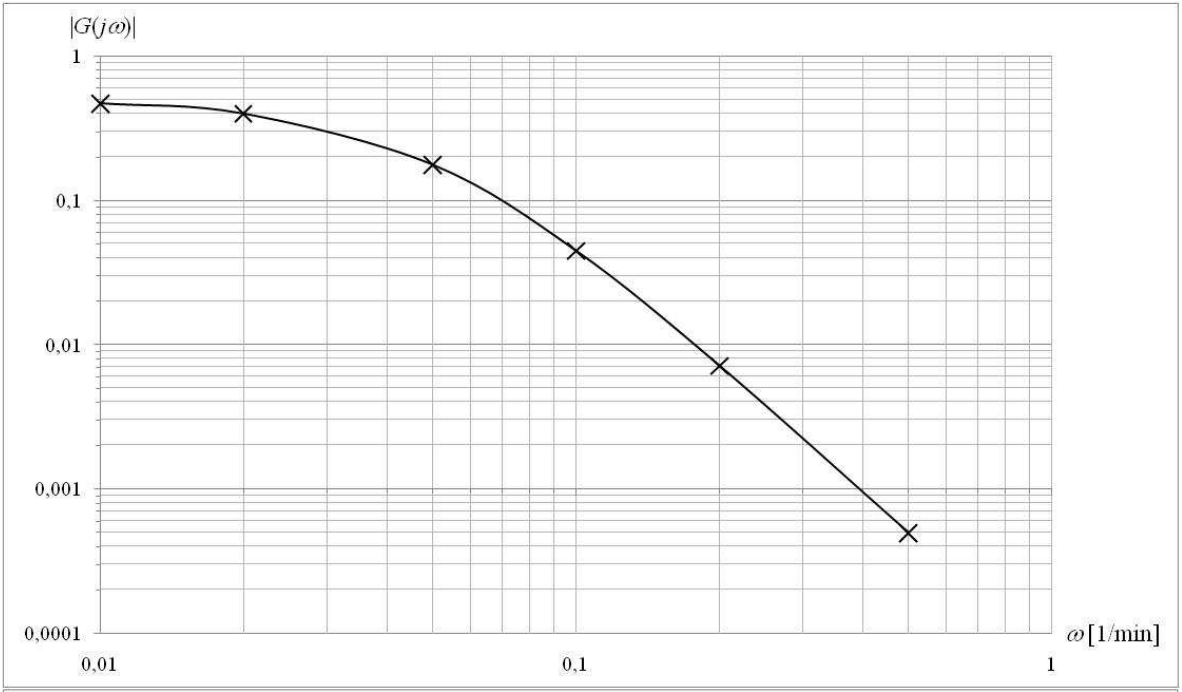
The Bode plots are shown below.

b. <u>Compute the tuning with P controller according to the Ziegler-Nichols table!</u>
The critical data belong to the frequency at which the phase shift is –180°. Reading the Bode plot we find that this is at ω_0 = 0.065 1/min. In that point the gain of the controlled section is 0.113. The critical gain is then

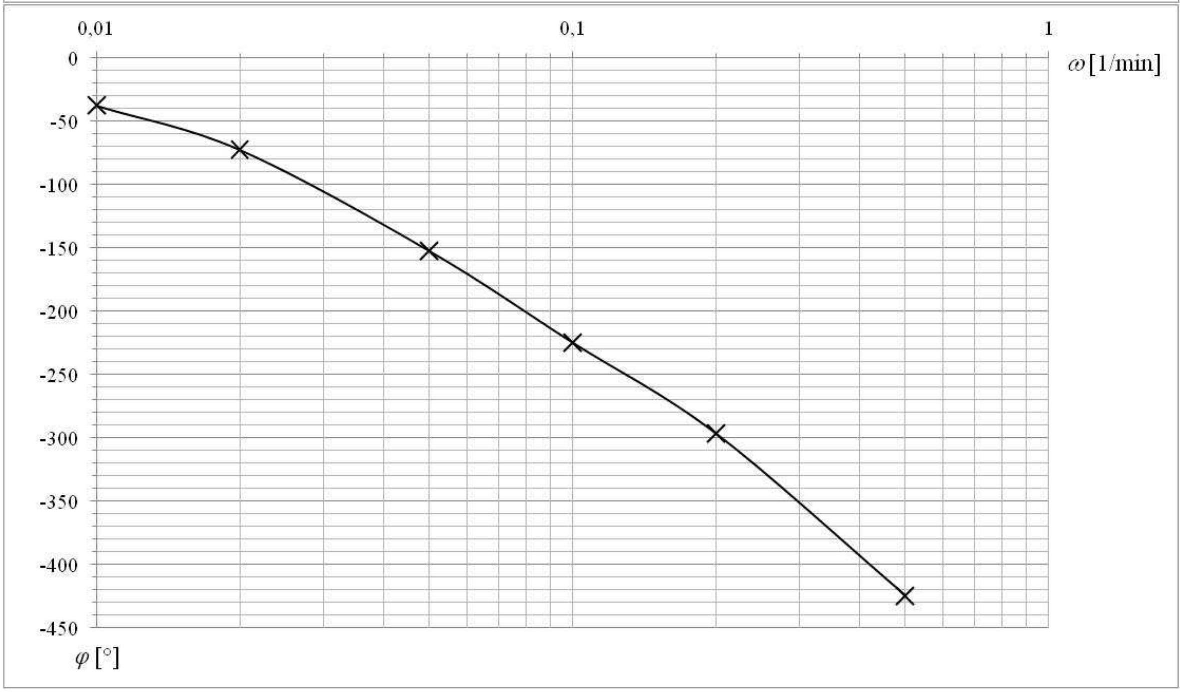
$$A_{C,crit} = \frac{1}{\lvert g(\omega)\rvert_{crit}} = \frac{1}{0.113} = 8.85$$

According to the Ziegler-Nichols table, the suggested tuning is:

$$A_C = 0.5 \cdot A_{C,crit} = 0.5 \cdot 8.85 = 4.42$$



$|G(j\omega)|$
1
0,1
0,01
0,001
0,0001
0,01
0,1
1
ω [1/min]



0,01
0,1
1
ω [1/min]
0
-50
-100
-150
-200
-250
-300
-350
-400
-450
φ [°]

c. Compute the phase margin of the closed loop!

First we are looking for the frequency at which the amplitude ratio of the open loop is 1:

$$\left|g^*(\omega)\right| = \left|g(\omega)\right| \cdot \left|g(\omega)\right|_P = 1$$

But |g| of a P-controller, at any frequency, equals its gain A_C: $\left|g(\omega)\right|_P = A_C = 4.42$

$$\left|g^*(\omega)\right| = \left|g(\omega)\right| \cdot 4.42 = 1$$

$$\left|g(\omega)\right| = \frac{1}{4.42} = 0.23$$

Either reading from the Bode plot or by iterative calculations we can find that the frequency at this point is $\omega = 0.04$ 1/min. The phase shift of the controlled section can be calculated, and is shown in the last column of the table above, it is $\varphi = -129.73°$. P controllers do not have any phase shift, thus this is the phase shift of the open loop as well.

Thus the phase margin is

$$PM = -129.73° - (-180°) = 50.27°$$

d. Compute the gain margin of the closed loop!

First we are looking for the frequency at which the phase shift is –180°. However, this is already found; $\omega_0 = 0.065$ 1/min, because the P controller does not have any phase shift.

The amplitude ratio of the controlled section at $\omega_0 = 0.065$ 1/min is

$$\left|g(\omega)\right|_{crit} = 0.113$$

The amplitude ratio of the P controller at any frequency is

$$\left|g(\omega)\right|_P = A_C = 4.42$$

The amplitude ratio of the open loop at $\omega_0 = 0.065$ 1/min is

$$\left|g^*(\omega)\right| = \left|g(\omega)\right|_{crit} \cdot \left|g(\omega)\right|_P = 0.113 \cdot 4.42 = 0.5$$

(Of course, since this is according to Ziegler-Nichols and no phase shift of the controller)

Thus the gain margin is

$$GM = \frac{1}{\left|g^*(\omega)\right|} = \frac{1}{0.5} = 2$$